

Self-Referential Cost-Functionals and Catastrophic Penalties: A Logostic Reading of “Resummation with Penalties”

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Resummation methods for divergent or asymptotic series are usually presented as purely technical tools for extracting reliable numbers from formally ill-behaved expansions. The recent work “Resummation with Penalties” by Simon Gluzman reframes this picture in a striking way. By combining generalized Borel transforms, self-similar iterated root approximants, and a family of cost-functionals with tunable penalty exponents, it exposes a rich optimization landscape in which catastrophic minima, scaling symmetries, and their deliberate violations all play crucial roles. The paper closes by explicitly invoking catastrophe theory and self-reference, and by suggesting that asymptotic series should be treated as imprints of complex systems rather than as merely formal objects. This article takes that invitation seriously. It proposes a “logostic” reading of cost-functional resummation, interpreting each optimization landscape as a one-dimensional slice through a higher-dimensional truth-favoring manifold and viewing iterated roots as self-referential, emergent structures. By examining how different penalty exponents move the system between smooth, scale-invariant minima and cusp-like, catastrophic optima, we connect Gluzman’s framework to broader ideas about complexity, edge-of-chaos behavior, and self-sustaining loops. We then sketch how this conceptual bridge can extend beyond traditional resummation, offering a toy model for the design of truth-favoring compression schemes in AI and other complex adaptive systems.

Keywords: iterated roots, Borel summation, cost-functionals, scaling violation, catastrophe theory, self-reference, emergent systems, complexity, logostics, asymptotic series, truth-favoring compression.

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1. Introduction

Resummation techniques occupy an ambiguous place in theoretical physics. On the one hand, they are presented as modest repair tools: ways of extracting finite, physically meaningful quantities from divergent or asymptotic series that appear in perturbation theory. On the other hand, whenever resummation works surprisingly well, it hints that the formal series is not just a defective approximation, but a partially encoded imprint of the underlying system's deeper structure. The move from “cleaning up a bad series” to “decoding the imprint of a complex system” is precisely where questions of emergence, self-reference, and catastrophe begin to appear.

Recent work on generalized Borel summation with iterated root approximants and cost-functionals with penalties makes this ambiguity explicit. Instead of treating resummation as a black box that takes a truncated power series and returns a number, it opens the black box and reveals a structured optimization landscape. In that landscape, symmetries such as scaling invariance can be preserved or deliberately broken, differentiability can be maintained or sacrificed, and catastrophic “crash landings” at sharp minima can outperform smooth, well-behaved approaches. It is in this richer setting that resummation starts to resemble a genuine theory of emergent behavior at the edge of chaos, rather than a purely technical trick.

This paper develops a logostic reading of that landscape. It treats cost-functionals, iterated roots, and catastrophic penalties as objects living on a larger manifold of truth-favoring structures, and it explores how the physics examples can serve as a toy model for more general questions about complexity and compression, including those arising in AI.

1.1 Resummation as more than a technical fix

In standard presentations, asymptotic series appear as a nuisance that can nonetheless be made useful. A quantity $f(x)$ is known only through a truncation

$$f_k(x) = \sum_{n=0}^k a_n x^n$$

that approximates $f(x)$ as x goes to 0, while the physically interesting behavior often lives at large x , where $f(x)$ behaves like a power law $f(x) \sim B x^\beta$. The classical question is how to extract the amplitude B from limited information about the coefficients a_n . Resummation methods are then introduced as a way to tame divergence, improve convergence, or analytically continue the series into the strong-coupling regime.

In this purely technical framing, the resummation device is judged by two criteria: numerical accuracy and stability across examples. The deeper structure of the method, and the shape of the optimization problems it entails, are secondary at best. However, once one inspects how

advanced schemes actually work, this modest narrative becomes inadequate. Resummation does not simply “fix” a defective series; it imposes a specific pattern of transformation and constraint that encodes hypotheses about the underlying system.

The more carefully one chooses the transformations, approximants, and optimization criteria, the more the resummation machinery begins to look like a model of emergent behavior itself. The truncated series is not just data; it is an imprint of a complex system. The resummation scheme is not just a numerical trick; it is a structured way of decoding that imprint. In this sense, resummation becomes a probe into how complexity and critical phenomena are written into asymptotic expansions, and how those writings can be read.

1.2 From generalized Borel transforms to cost-functionals with penalties

The work this paper engages with begins from a generalized Borel transform. Instead of directly manipulating the original coefficients a_n , one considers transformed coefficients

$$b_n = a_n (\Gamma(1 + n))^p$$

that depend on a real control parameter p . This defines a family of transformed truncations

$$b_k(x, p) = \sum_{n=0}^k b_n x^n,$$

each more or less amenable to summation depending on the choice of p . The parameter p is not merely a nuisance; it is a knob that interpolates between different ways of reweighting the original information, changing how the series encodes the large- x amplitude B .

To carry this information toward the strong-coupling regime, iterated root approximants are introduced. These approximants are defined recursively so that, order by order, their small- x expansion matches the transformed series, while their large- x behavior has the desired power-law form with exponent β . This two-sided constraint—matching the local expansion and enforcing the asymptotic structure—imposes a highly nontrivial shape on the approximants.

The presence of the control parameter p means that, for each truncation order k , there is an entire one-parameter family of candidate amplitudes $B_k(p)$. Choosing among them is nontrivial, especially when simple optimization conditions such as minimal difference between successive orders or minimal derivative with respect to p yield multiple solutions. To resolve this non-uniqueness, one moves from pointwise optimization conditions to global cost-functionals: scalar functionals $F(p, m)$ that incorporate differences between orders, derivatives with respect to p , and penalties for deviation from reference amplitudes.

The penalty terms are the crucial new element. They make explicit a trade-off between smoothness, stability, and fidelity to chosen reference values. By adjusting the functional form

and exponent m in these penalties, one can move between scale-invariant, differentiable cost-functionals and non-homogeneous, non-differentiable ones that induce catastrophic behavior.

1.3 Emergent behavior, catastrophe, and self-reference in resummation

Once cost-functionals with penalties are introduced, the optimization landscape over p acquires a rich structure. For certain exponents m , the cost-functional is homogeneous in the amplitudes and respects scaling symmetry: scaling all amplitudes by a common factor simply rescales the cost, leaving the location of minima unchanged. For other exponents, this homogeneity is broken; the cost-functional no longer respects scaling, and the symmetry is softly or strongly violated.

This change in symmetry properties is accompanied by a change in differentiability. For particular choices of m , the cost remains smooth, and the approach to the minimum is gradual. For other choices, the cost becomes non-differentiable at the optimum; the first derivative exhibits an essential singularity, and the optimum is reached in a catastrophic, crash-landing fashion. The parameter m thus controls both symmetry and smoothness, and it does so in a way that can dramatically alter the behavior of the optimization.

At the same time, the iterated root approximants themselves are self-referential structures. Their recursive definition preserves the functional form at each order, with parameters inherited from lower orders and updated in light of new coefficients. This recursion is not a simple hierarchy; it is a loop in which the same pattern is applied repeatedly, with the resulting amplitudes feeding back into the optimization of p through the cost-functional. The system as a whole exhibits emergent behavior: sharp transitions in the optimum as m varies, robust convergence in some examples but not others, and non-uniqueness of solutions that echoes indeterminism and edge-of-chaos phenomena in complex systems.

Seen in this light, resummation becomes a laboratory for catastrophe theory and self-reference. The cost-functional defines a one-dimensional catastrophe surface over p , with folds and cusps induced by the penalties. The iterated roots define a self-referential loop that gives the system memory across orders of perturbation theory. The asymptotic amplitude B emerges as a fixed point of this self-referential, catastrophically shaped optimization process.

1.4 Aim and structure of this paper

The aim of this paper is to articulate a logostic interpretation of cost-functional resummation with penalties. By “logostic” we mean an approach that treats the optimization landscape, and the structures that inhabit it, as living on a manifold shaped by a direction toward truth, or τ ,

rather than as neutral numerical machinery. This interpretation seeks to connect concrete details of generalized Borel transforms, iterated root approximants, and penalty-based cost-functionals with broader ideas about emergence, self-reference, catastrophe, and truth-favoring compression.

The paper proceeds as follows. Section 2 describes truncated asymptotic series, generalized Borel transforms, and iterated root approximants, emphasizing their role as self-similar, self-referential structures that compress information about critical behavior into a small set of parameters. Section 3 introduces cost-functionals based on amplitudes and derivatives, explains the role of homogeneity and scaling symmetry, and contrasts scale-respecting functionals with those that deliberately violate symmetry.

Section 4 explores how different penalty exponents m shape the optimization landscape over p , producing smooth or catastrophic approaches to the optimum and inviting a reading in terms of catastrophe theory. Section 5 surveys the performance of these schemes across a range of physical examples, highlighting the tension between robustness and optimality. Section 6 develops the logostic reading explicitly, treating cost-functionals as slices through a truth-favoring manifold and catastrophic minima as sharp bends in a truth-potential.

Section 7 focuses on self-reference, interpreting iterated roots as self-sustaining loops or Markov membranes, and Section 8 situates the non-uniqueness and catastrophic behavior at an effective edge of chaos. Section 9 extends the analogy to AI, treating resummation as a toy model for truth-favoring compression and alignment. Section 10 concludes with open questions and prospects for bridging asymptotic analysis, complex systems, and AI through a shared language of logostics.

2. Asymptotic Imprints and Self-Referential Iterated Roots

Asymptotic expansions arise when a physical or mathematical system is probed in a particular regime, often at small values of a dimensionless variable such as a coupling or scaled distance. The result is a truncated series that captures local structure but appears to be almost useless in the regime of real interest, which may lie at the opposite extreme of that variable. The resummation framework under discussion begins from this familiar situation, but interprets the truncated series as a compressed imprint of the underlying complex system rather than as a defective approximation.

In this section, we unpack that perspective. First, we regard truncated asymptotic series as fingerprints of complex systems, with their coefficients encoding nontrivial information about critical behavior. Then we introduce generalized Borel transforms with a real control parameter p , which reweight these coefficients and open a one-parameter family of possible

resummations. Next, we describe iterated root approximants, which enforce both local and asymptotic constraints and possess a self-similar structure across orders. This leads naturally to a reading of the recursive construction as self-referential: parameters are passed from order to order in a loop that preserves functional form. Finally, we interpret the asymptotic amplitudes produced by this machinery as compressed summaries of critical behavior, taking the role of scalar invariants that encode global structure in a single number.

2.1 Truncated asymptotic series as fingerprints of complex systems

Many quantities of interest in field theory, statistical mechanics, and quantum mechanics are not known in closed form. Instead, what is available is a formal expansion around some limit, typically expressed as

$$f_k(x) = \sum_{n=0}^k a_n x^n,$$

valid as x tends to zero. Here x may represent an inverse coupling, a scaled field, a density, or some other dimensionless parameter. The coefficients a_n are produced by perturbation theory, diagrammatic expansions, or other systematic procedures. For small x , the truncation $f_k(x)$ can be an excellent approximation to the true function $f(x)$. For large x , it usually diverges or becomes meaningless.

From a naive standpoint, such a series is a local object in x , blind to the critical behavior that emerges at large x , where $f(x)$ often behaves as a power law

$$f(x) \sim B x^{\beta},$$

with a critical exponent β and amplitude B that encode universal and non-universal information, respectively. The surprising power of resummation methods suggests that this view is incomplete. Even though the series is derived from local information, the pattern of coefficients a_n carries global constraints inherited from the underlying theory. It is not arbitrary data; it is a fingerprint of the complex system that generated it.

This fingerprint is shaped by symmetries, conservation laws, analyticity properties, and renormalization-group structure. The coefficient growth pattern, alternating signs, and ratios all hint at singularities in the complex plane and at the asymptotic scaling of the function. When a resummation method succeeds in reconstructing the large- x amplitude B from the first few coefficients of $f_k(x)$, it is exploiting those hidden constraints. The truncated asymptotic series then becomes a compressed imprint of the full system, from which critical behavior can, in favorable cases, be reconstructed.

2.2 Generalized Borel transforms and the control parameter p

The generalized Borel transform used in the cost-functional framework begins by reweighting the original coefficients a_n via the gamma function. One defines

$$b_n(p) = a_n (\Gamma(1 + n))^p,$$

and constructs the transformed truncation

$$b_k(x, p) = \sum_{n=0}^k b_n(p) x^n.$$

The familiar Borel transform corresponds to a special choice of p , but the generalized form allows p to vary over the reals. This introduces a control parameter into the resummation process. For each value of p , the coefficients are rescaled in a way that modifies the effective contribution of higher-order terms.

Heuristically, the factor $(\Gamma(1 + n))^p$ behaves like $(n!)^p$ for large n . For positive p , higher-order terms are strongly reweighted, while for negative p they are relatively suppressed. This reweighting changes the balance between local and global information encoded in the series. Some choices of p may make the transformed series more amenable to subsequent resummation; others may distort it in ways that obscure the asymptotic structure.

The key point is that p is not just a nuisance parameter that must be eliminated. It defines a one-parameter family of possible resummations. The true asymptotic amplitude B is independent of p , but any finite-order approximation $B_k(p)$ will depend on it. This dependence creates an optimization problem: find the value or values of p that best reconcile the transformed truncation with the asymptotic behavior and with other constraints. The cost-functional approach formalizes this optimization, but already at this stage, the presence of p indicates that we are dealing with a landscape of possibilities, not a unique mechanical transformation.

2.3 Iterated root approximants as self-similar structures

To move from the transformed series $b_k(x, p)$ to an approximation that captures both small- x and large- x behavior, the framework employs iterated root approximants. These are defined recursively, starting from the lowest order and building up step by step. At each step, the approximant $C_k^*(x, p)$ is constructed so that its Taylor expansion around $x = 0$ matches the transformed series up to order k , while its large- x behavior is constrained to follow a power law with exponent β .

In schematic form, the recursion can be written as

$$C_0^*(x, p) = a_0,$$

$$C_k^*(x, p) = (C_{k-1}^*(x, p))^{\{k / \beta\}} + A_k(p) x^k)^{\{\beta / k\}},$$

for $k = 1, 2, 3$, and so on, where the parameters $A_k(p)$ are determined by matching the series expansion of $C_k^*(x, p)$ to $b_k(x, p)$. The precise algebraic forms of $A_k(p)$ are obtained uniquely from this matching condition. What matters for the present discussion is the structural pattern of the recursion.

Each new approximant is built from the previous one by raising it to a power, adding a new term proportional to x^k , and then raising the whole expression to another power. The functional form is preserved across orders: every $C_k^*(x, p)$ is an iterated root of a sum of powers of x , with parameters inherited from lower orders. The structure is self-similar. As k increases, new layers are added to the nested root, but the basic pattern remains the same.

This self-similarity is what makes the iterated root scheme a natural candidate for interpreting the asymptotic behavior. As x tends to infinity, the inner structure of the nested root becomes dominated by the highest-order term, and the approximant behaves as

$$C_k^*(x, p) \sim c_k(p) x^{\beta},$$

where $c_k(p)$ is the emergent amplitude at order k . In this way, the iterated root structure acts as a bridge between local series data and global power-law behavior. It enforces a specific pattern of emergent scaling that is repeated across orders and controlled by the parameters $A_k(p)$ and the exponent β .

2.4 Self-reference in the recursion: parameters passed order to order

The recursive construction of iterated root approximants is not just self-similar; it is self-referential. Each new approximant $C_k^*(x, p)$ is defined in terms of its predecessor $C_{k-1}^*(x, p)$, which already encodes information about all lower orders. The parameters $A_k(p)$ are chosen so that the small- x expansion of $C_k^*(x, p)$ matches $b_k(x, p)$, but this choice necessarily depends on the existing structure of $C_{k-1}^*(x, p)$. In this way, the recursion passes information forward and backward between orders.

At order k , the approximant has access to the entire history of lower-order coefficients through C_{k-1}^* , and it integrates the new coefficient associated with x^k via $A_k(p)$. The same functional template is reused: a power of the previous approximant, plus a new polynomial term, all raised to a suitable power. This reuse of form, coupled with updating of parameters, is

characteristic of self-referential systems. The system refers to its own previous state when constructing the next one.

This self-reference is not an incidental feature. It is central to how the approximants stabilize and converge. The behavior of $C_k(x, p)$ at large x depends on the interplay between the inherited structure in C_{k-1} and the newly added term. Because the functional form remains the same, the asymptotic exponent β is preserved while the amplitude $c_k(p)$ is updated. The sequence of amplitudes $c_1(p), c_2(p), \dots, c_k(p)$ can then be interpreted as the orbit of a self-referential map that acts on the space of candidate amplitudes.

When this sequence converges, or at least stabilizes in an approximate sense, the limiting amplitude can be regarded as a fixed point of the self-referential loop. The fact that a nontrivial fixed point emerges from a finite number of perturbative coefficients is what gives the method its power. The iterated roots do not simply sum a series; they implement a self-referential construction in which each step depends on the entire previous structure.

2.5 Asymptotic amplitudes as compressed summaries of critical behavior

The endpoint of this construction is the asymptotic amplitude B , which appears in the large- x behavior

$$f(x) \sim B x^\beta$$

of the original function. Within the generalized Borel and iterated root framework, one considers the sequence of approximate amplitudes $B_k(p)$, obtained by mapping $c_k(p)$ back through the inverse of the generalized Borel transform. For each order k and each choice of p , $B_k(p)$ is a candidate for the true amplitude B . The cost-functional approach will later select among these candidates, but already at this stage, $B_k(p)$ has a special role.

From one perspective, $B_k(p)$ is just a scalar output of a resummation algorithm. From another, it is a compressed summary of critical behavior. The entire vector of coefficients $a_0, a_1, \dots, a_k$, together with the choice of transformation parameter p and the iterated root structure, is mapped to a single real number $B_k(p)$. This number encodes how the system behaves in a strongly different regime from the one where the original data were collected. It is a scalar invariant extracted from a higher-dimensional imprint.

The distinction between the critical exponent β and the amplitude B is instructive here. In many cases, β is known from symmetry arguments, scaling theory, or renormalization-group analysis. It captures universal information that does not depend sensitively on microscopic details. The amplitude B , by contrast, is often non-universal. It depends on specific features of the system, such as boundary conditions, microscopic interactions, or normalization choices.

Recovering B from a truncated series therefore amounts to recovering non-universal, system-specific information from a universal-looking imprint.

In this sense, the asymptotic amplitude B plays a role analogous to a model parameter inferred from data in other contexts. It is the quantity that must be estimated from incomplete information, subject to constraints imposed by known structure. The generalized Borel plus iterated root machinery, with its self-referential recursion and control parameter p , provides one way to perform such an estimation. When it works, the result is a highly compressed summary of critical behavior: a single number that captures, in distilled form, the outcome of complex interactions that are only partially visible in the original truncation.

Viewed this way, asymptotic amplitudes are not just by-products of resummation; they are the central objects. They represent the success or failure of the method to decode the fingerprint left by the complex system in its asymptotic series. The cost-functionals and penalties introduced later in the framework will act directly on sequences of such amplitudes, treating them as the primary carriers of information about the underlying critical behavior.

3. Cost-Functionals, Scaling Symmetry, and Its Violations

The move from pointwise optimization conditions to explicit cost-functionals is the conceptual heart of the resummation-with-penalties framework. Instead of solving separate equations that enforce, for example, minimal difference between successive amplitudes and minimal sensitivity with respect to the control parameter p , one combines these desiderata into a single scalar function $F(p, m)$ to be minimized. This scalar function is constructed from the amplitudes produced by the iterated root approximants, their derivatives with respect to p , and penalty terms that tie them to reference values.

In this section, we examine how these cost-functionals are built, and why their detailed form matters. We begin by describing how the amplitudes $B_k(p)$, their differences across orders, and their derivatives enter the construction. We then discuss the special role of homogeneous, scale-respecting functionals with exponent $m = 2$, which preserve scaling symmetry in the amplitudes. Next, we analyze non-homogeneous penalties that deliberately violate scaling symmetry, and we treat this violation as a controlled deformation of the optimization landscape. We then turn to differentiability, emphasizing how different choices of m lead to smooth or non-smooth costs, and how essential singularities in the first derivatives introduce catastrophic behavior. Finally, we compare these cost-functionals to more familiar Tikhonov and ridge-type regularization schemes, highlighting the conceptual differences that arise when penalties act on amplitudes rather than directly on the control parameter.

3.1 Constructing cost-functionals from amplitudes and derivatives

The basic building blocks of the cost-functionals are the approximate amplitudes $B_k(p)$ obtained from iterated root approximants and the generalized Borel transform at order k , together with their derivatives with respect to the control parameter p . The simplest quantities that probe stability and consistency across orders are the difference between successive amplitudes,

$$B_k(p) - B_{k-1}(p),$$

and the derivative of $B_{k-1}(p)$ with respect to p ,

$$d B_{k-1}(p) / d p.$$

The difference measures how much the amplitude changes when one more order of the perturbative series is included; small differences are a sign of convergence or at least stability. The derivative measures sensitivity to the choice of p ; a small derivative indicates that the amplitude is relatively insensitive to small changes in p , a notion of minimal sensitivity.

These ingredients suggest quadratic contributions to the cost, such as

$$|B_k(p) - B_{k-1}(p)|^2,$$

and

$$|d B_{k-1}(p) / d p|^2,$$

possibly combined with weights that allocate different importance to matching across orders versus minimal sensitivity. A natural first step is to define a baseline cost-functional that is a weighted sum of these two terms. This captures the intuitive idea that a good choice of p should make successive amplitudes similar and should lie in a region where the amplitude is locally flat as a function of p .

To guide the resummation toward a specific reference behavior, additional penalties are introduced. One class of reference values comes from a special choice of the Borel parameter, typically $p = 1$, which is treated as a distinguished baseline. Another class comes from discrete approximations b_k^* constructed from a finite set of discrete values of p . Penalty terms of the form

$$|B_k(p) - B_k(1)|^m$$

or

$$|B_k(p) - b_k^*|^m$$

are then added to the baseline cost, where m is a positive exponent that tunes the shape of the penalty. The full cost-functional thus becomes a sum of quadratic stability terms and a penalty term raised to a power m , with an overall weight λ controlling the relative importance of stability and sensitivity contributions.

The optimization problem is then to find p that minimizes $F(p, m)$ for a given m . The resulting optimal p , denoted $p_{\text{opt}}(m)$, defines a sequence of amplitudes

$$B_k^*(m) = B_k(p_{\text{opt}}(m)),$$

which will be analyzed across k to assess convergence and robustness. In this way, the cost-functional compresses multiple desiderata into a single scalar criterion, and the choice of exponent m becomes a central design parameter.

3.2 Homogeneous, scale-respecting functionals with $m = 2$

When the penalty exponent m is set to 2, the cost-functionals acquire a special property: homogeneity of degree two in the amplitudes. Concretely, suppose that all amplitudes entering the cost-functional, such as $B_k(p)$, $B_{\{k-1\}}(p)$, and b_k^* , are scaled by a common multiplicative factor γ . Then, for $m = 2$, each term in the cost-functional scales as γ squared. The stability terms are obviously quadratic in amplitudes, and the penalty term $|B_k(p) - \text{reference}|^2$ is likewise quadratic. As a result, the entire cost-functional satisfies

$$F(\gamma B_k, \gamma B_{\{k-1\}}, \gamma b_k^*) = \gamma^2 F(B_k, B_{\{k-1\}}, b_k^*).$$

This homogeneity implies that the location of minima in p -space is invariant under uniform rescaling of all amplitudes. Rescaling simply rescales the value of the cost at each p , but it does not change which p minimizes the cost. In physical terms, the functional is sensitive only to relative differences and derivatives of amplitudes, not to their overall scale. This is what is meant by scale-respecting behavior.

The preservation of scaling symmetry is more than a technical convenience. Many physical systems admit overall rescalings of observables that do not change their qualitative behavior. In the context of amplitude extraction, multiplying all amplitudes by a common factor does not alter the underlying structure of the optimization problem; it merely changes units. By constructing cost-functionals that are homogeneous of degree two, one respects this symmetry and avoids introducing artificial preferences for particular scales.

Moreover, quadratic penalties have favorable analytic properties. They are smooth and differentiable everywhere, they yield linear first-order conditions for stationarity, and their influence grows rapidly with the size of deviations, which is often desirable for suppressing large

errors. The combination of homogeneity, symmetry preservation, and analytic simplicity makes the $m = 2$ case a natural baseline.

However, homogeneity also limits flexibility. A scale-respecting cost-functional cannot distinguish between configurations that differ only by an overall rescaling; it can only evaluate relative structure. If there are situations where absolute scales matter, or where breaking scaling symmetry improves robustness, one may need to go beyond the $m = 2$ case. This motivates the introduction of non-homogeneous penalties with m not equal to 2.

3.3 Non-homogeneous penalties and controlled scaling violation

When the penalty exponent m differs from 2, the cost-functionals lose their homogeneity of degree two, and with it, strict scaling symmetry. If amplitudes are rescaled by γ , the stability terms still scale as γ^2 , but the penalty term now scales as γ^m . As a result, the overall cost-functional scales in a mixed way, and the location of its minima in p -space can shift under rescaling.

This deliberate departure from homogeneity is not a bug; it is a feature. It introduces a controlled scaling violation into the optimization process. The magnitude of this violation depends on how far m is from 2. For exponents $m = 3/2$ or $m = 5/2$, the deviation from quadratic scaling is relatively modest, and one can speak of weak scaling violation. For exponents $m = 1$ or $m = 1/2$, the violation is more pronounced, and the qualitative behavior of the optimization landscape changes.

The physical intuition behind this choice is that there may be situations where absolute deviations from reference amplitudes matter more than relative differences, especially when the underlying Monte Carlo or exact results used as references are quoted with finite accuracy. A non-homogeneous penalty can amplify or suppress certain regions of amplitude space in a way that would be impossible under strict homogeneity. In particular, penalties with m less than 2 can be more forgiving of moderate deviations while still exerting strong influence near the optimum.

From a geometric perspective, varying m deforms the cost surface over p . For m close to 2, the surface resembles a gently curved bowl with a well-defined minimum. As m moves away from 2, the curvature near the minimum and the shape of the approach to that minimum are altered. When m is lowered to 1 or $1/2$, the cost-functional can develop cusp-like behavior at the minimum: the surface becomes sharply peaked, and the gradient changes rapidly. This is the beginning of catastrophic behavior, in the sense that small changes in p can produce large changes in the cost near the optimum.

In this framework, scaling violation is not a random artifact but a tunable parameter. Choosing m allows one to adjust how strongly the cost-functional reacts to absolute scales and how sharply it enforces closeness to reference amplitudes. The empirical result, borne out across multiple physical examples, is that certain non-homogeneous choices of m can yield more robust or more accurate amplitude estimates than the homogeneous, scale-respecting case. This motivates a more systematic exploration of the trade-offs induced by m .

3.4 Differentiability, essential singularities, and non-smooth costs

Beyond scaling, the exponent m also governs differentiability. When m is greater than or equal to 2, the penalty term $|B_k(p) - \text{reference}|^m$ is not only continuous but also differentiable with a continuous derivative. The total cost-functional $F(p, m)$ inherits this smoothness, and standard calculus-based optimization techniques can be used. The approach to the minimum is smooth, and the gradient vanishes in a regular way.

For exponents between 1 and 2, such as $m = 3/2$ or $m = 5/2$, the situation remains relatively benign. The penalty term is still differentiable everywhere, but its derivative behaves differently near zero. The gradient grows or decays with a fractional power of the deviation, altering the sharpness of the minimum but not introducing true singularities. The cost surface is still smooth, and the optimum is approached steadily rather than abruptly.

The most interesting behavior arises for m equal to 1 or $1/2$. In these cases, the penalty term is continuous but not differentiable in the usual sense at the point where the deviation vanishes. For $m = 1$, the penalty is proportional to the absolute value of the deviation. Its derivative with respect to the amplitude jumps discontinuously at zero, producing a cusp in the cost surface. For $m = 1/2$, the situation is more severe: the derivative diverges like the reciprocal of the square root of the deviation. This divergence translates into an essential singularity in the derivative of the cost-functional with respect to p at the optimum, provided that the amplitude depends smoothly on p .

These non-smooth behaviors give rise to what is described as catastrophic approach to the optimum. As p approaches p_{opt} , the cost can drop sharply, with the gradient becoming very large in magnitude near the minimum or changing sign abruptly. The optimization landscape develops cusp-like or fold-like features, reminiscent of canonical examples in catastrophe theory. The path to the optimum is no longer a gentle descent; it is more like a crash landing.

From a numerical standpoint, non-smooth costs can be both a challenge and an advantage. They may cause difficulties for gradient-based methods, but they can also sharpen the localization of the optimum, making it less sensitive to small perturbations. From a conceptual standpoint, the appearance of essential singularities in the derivative is significant. It indicates

that small changes in amplitudes can lead to disproportionate changes in the effective force driving p toward the optimum. This is a hallmark of systems poised near the edge of qualitative change, which is precisely the territory that catastrophe theory and complexity studies inhabit.

3.5 Comparing cost-functionals to Tikhonov and ridge-type regularization

The cost-functionals built from amplitudes and their derivatives invite comparison with classical regularization schemes, particularly Tikhonov regularization and ridge-type penalties. In these well-known frameworks, one typically minimizes a functional of the form

data misfit + regularization term,

where the regularization term is often quadratic in the parameters, such as a squared norm. The purpose is to stabilize ill-posed problems, reduce overfitting, and enforce smoothness or smallness in the parameter space.

In the resummation context, a Tikhonov-like cost-functional can be constructed by combining the squared difference between successive amplitudes, the squared derivative with respect to p , and a quadratic penalty on the deviation of p from a preferred value, such as $p = 1$. The form

$$|B_k(p) - B_{k-1}(p)|^2 + |dB_{k-1}(p)/dp|^2 + (p - 1)^2$$

is a direct analogue: the first two terms express consistency and minimal sensitivity, while the last term is a ridge penalty that pulls p toward a reference. This functional is homogeneous of degree two in the amplitudes, except for the last term, which involves only p . The presence of $(p - 1)^2$ thus breaks amplitude scaling symmetry, but in a different way than the non-homogeneous amplitude penalties discussed earlier.

The conceptual difference is that amplitude-based cost-functionals treat the amplitudes themselves as the primary objects of regularization. Their penalties involve distances in amplitude space, either between successive orders or between amplitudes and reference values. Tikhonov-like functionals, by contrast, place a direct penalty on the parameter p , independent of how it affects the amplitudes. The former view sees the optimization problem as one of navigating a landscape of amplitudes; the latter sees it as one of choosing a regularized parameter.

Empirically, amplitude-based functionals with carefully chosen exponents m often outperform the simpler Tikhonov-like functional, especially in problems where the relationship between p and the amplitude is highly nonlinear. The ridge penalty $(p - 1)^2$ is blind to this nonlinearity; it favors values of p close to 1, regardless of how that choice affects the sequence of amplitudes. In contrast, amplitude-based penalties tie the optimization tightly to the actual quantities of

interest, and their non-homogeneous, non-smooth variants can exploit subtle structure in the amplitude landscape.

From a broader perspective, this comparison highlights a shift of emphasis. Rather than stabilizing the control parameter directly, the cost-functionals in this framework regularize the emergent quantities (amplitudes) and accept that the optimal p may lie far from naive expectations. The penalty exponent m becomes a design parameter that shapes the effective regularization in amplitude space, breaking or preserving scaling symmetry as needed. This shift is central to the logostic reading, in which the cost-functional is not just a technical device but a surface in a higher-dimensional manifold that encodes how the system is steered toward truth-favoring amplitudes.

4. Catastrophic Penalties as One-Dimensional Catastrophe Surfaces

Once cost-functionals with tunable penalty exponents are introduced, the resummation problem acquires a natural geometric picture. For a fixed physical problem and truncation order k , each choice of exponent m defines a scalar function of one variable,

$$F(p, m),$$

where p is the generalized Borel parameter. Plotting F as a function of p gives a one-dimensional curve; letting m vary turns this into a family of curves, or a surface over the (p, m) plane. The striking observation in the underlying work is that this surface is not always gently curved. For some values of m , it develops sharp features: narrow minima, cusps, and abrupt transitions between competing minima. These are “catastrophe-like” in the sense of catastrophe theory: small changes in control parameters can produce sudden changes in the optimal p .

In this section, we analyze how the exponent m shapes the optimization landscape, how cusp- and fold-like features appear in the one-dimensional p coordinate, and how weak versus strong scaling violation corresponds to smooth versus catastrophic approaches to the optimum. We then outline heuristic links to Thom’s catastrophe theory and interpret non-uniqueness of solutions as an imprint of underlying chaotic or edge-of-chaos structure in the extrapolation problem.

4.1 The role of the exponent m in shaping the optimization landscape

The exponent m enters the cost-functional through penalty terms of the form

$$|B_k(p) - \text{reference}|^m,$$

where “reference” stands for either a distinguished Borel amplitude such as $B_k(1)$, or a discrete composite amplitude such as b_k^* . For a given physical problem and order k , the function $B_k(p)$ is determined by the iterated root construction, and the cost-functional is assembled by combining amplitude differences, sensitivity terms in dB/dp , and the penalty.

The simplest way to see the influence of m is to imagine that, near a candidate optimum, the amplitude difference

$$\delta(p) = B_k(p) - \text{reference}$$

behaves approximately linearly in $p - p_0$. Then, near p_0 , the penalty term behaves like

$$|\delta(p)|^m \sim |p - p_0|^m.$$

Superimposed on this penalty are smoother quadratic terms coming from $|B_k(p) - B_{k-1}(p)|^2$ and $|dB_{k-1}(p)/dp|^2$. For $m = 2$, the penalty is quadratic in $p - p_0$, and the total cost near p_0 is essentially a sum of quadratic terms. For $m \neq 2$, the local curvature and slope of the cost near p_0 change in a systematic way.

When $m > 2$, the penalty is relatively flat near p_0 and grows faster at larger deviations. When $1 < m < 2$, the penalty is steeper near p_0 but grows more slowly farther away. When $m = 1$, the penalty is a V-shaped absolute value. When $m = 1/2$, its derivative diverges at zero, producing an extremely sharp valley. Therefore, by tuning m , one reshapes the local geometry of $F(p, m)$ around its minima. This reshaping can turn a broad basin into a narrow well, or vice versa, and it can determine whether the approach to the minimum is gradual or abrupt.

Because the rest of the cost-functional depends on $B_k(p)$ and its derivative, which are themselves nonlinear and problem-dependent, the influence of m is not purely local. Changing m can cause previously subdominant minima to become global, or vice versa, and it can qualitatively alter the number and location of critical points of F . In this sense, m is a genuine control parameter for the optimization landscape, not merely a tuning knob for the steepness of a single well.

4.2 Cusps, folds, and “crash landing” minima in one parameter p

The numerical examples in the underlying work show several distinct patterns in $F(p, m)$ as m varies. For values of m near 2, plots of $F(p, m)$ as a function of p often exhibit broad, smooth minima, with gentle slopes on either side. The function is differentiable, the derivative crosses zero cleanly, and the minimum is approached in a gradual way. This is the smooth, non-catastrophic regime.

As m decreases to values such as $3/2$ or $5/2$, the minima start to sharpen. The curvature near the bottom of the well increases, and in some cases competing shallow minima begin to appear at nearby values of p . The surface looks more like a slightly folded sheet than a perfect parabola. These features signal that the optimization is becoming more sensitive to p and that the landscape is closer to exhibiting catastrophe-like behavior.

The most dramatic behavior emerges for $m = 1$ and $m = 1/2$. In these cases, the cost-functional can develop cusp-like structures in p . Near the optimum p_{opt} , the graph of $F(p, m)$ may have a kink (for $m = 1$) or a very steep, almost vertical descent (for $m = 1/2$). As p approaches p_{opt} from either side, the cost drops rapidly, and the effective “force” driving p toward the minimum grows large. The derivative dF/dp may be discontinuous at p_{opt} or may diverge there, depending on the underlying behavior of $B_k(p)$.

When viewed as a function of p alone, this is still a one-dimensional object. Yet its shape resembles the cross-section of a higher-dimensional catastrophe surface. Folds appear where two local minima are separated by a narrow barrier or where a single minimum is pinched into a sharper shape. Cusps appear where the cost surface changes direction abruptly. The term “crash landing” captures the phenomenology: the system evolves along p until it suddenly falls into a narrow, sharply defined minimum, with little room for ambiguity about the optimal value once the descent is complete.

Such crash-landed minima can be advantageous for selecting a unique p in the presence of multiple possible solutions. Once the system is near the cusp, the cost-functional strongly favors the optimum, and alternative choices of p become significantly more costly. At the same time, the sharpness of the feature signals that the optimization landscape is poised near a structural change: small shifts in the problem data or in m could create or annihilate minima, or move them abruptly.

4.3 Weak versus strong scaling violation: smooth versus catastrophic approach

Scaling violation, introduced by choosing m different from 2, can be graded as weak or strong. When m is close to 2, such as $m = 3/2$ or $m = 5/2$, the penalty term departs only modestly from quadratic behavior. The cost-functional is no longer exactly homogeneous of degree two in the amplitudes, but the distortions introduced are relatively mild. In these cases, the optimization landscape over p tends to remain smooth, with minima that sharpen but do not become singular. The approach to the optimum is still gradual, albeit with altered curvature.

This is the weak scaling violation regime. It is characterized by a shift in the relative importance of amplitude deviations versus stability and sensitivity terms, and by a moderate reshaping of the cost surface. Empirically, weak scaling violation often improves robustness and accuracy without fundamentally changing the qualitative nature of the optimization.

In contrast, strong scaling violation arises for m near 1 or $1/2$. Here, the penalty term dominates the local structure of the cost-functional near small deviations, and its non-smooth or singular derivative behavior controls the shape of $F(p, m)$. The landscape can develop sharp valleys and cusp-like features, and the gradient with respect to p can become very large near the optimum. This leads to the catastrophic approach: the system moves through p -space relatively gently until it enters a region where the cost drops steeply and rapidly, forcing a crash landing at p_{opt} .

From the point of view of symmetry, strong scaling violation breaks the scale invariance of the amplitudes in a pronounced way. The optimization no longer respects rescaling as a benign transformation; absolute amplitude differences, rather than just relative structure, now strongly influence the location and sharpness of the minimum. From the point of view of numerical behavior, this can either improve or degrade performance, depending on the problem. In several of the examples analyzed, strong scaling violation produced better agreement with known amplitudes, but at the price of a more delicate landscape.

In summary, weak scaling violation corresponds to a smooth deformation of the cost surface, while strong scaling violation can induce catastrophic features. The exponent m thus not only tunes scaling properties but also controls whether the optimization problem is gentle or poised near a qualitative transition.

4.4 Heuristic links to Thom's catastrophe theory

Catastrophe theory, in the sense introduced by Thom and others, deals with how the equilibria of a system described by a potential function depend on control parameters. The canonical examples involve simple potentials in one or two behavior variables and a small number of control parameters, leading to standard forms such as the cusp, fold, and swallowtail

catastrophes. The key idea is that small changes in control parameters can cause sudden jumps in equilibria when the underlying potential surface has folds or cusps.

The cost-functional $F(p, m)$ can be viewed, heuristically, as a potential function in one behavior variable p , with several implicit control parameters: m , the weight λ between stability and sensitivity terms, the truncation order k , and problem-specific quantities such as the known exponent β and reference amplitudes. For fixed k and physical problem, we can think of the family $F(p; m, \lambda, \dots)$ as a section of a higher-dimensional catastrophe surface over the control parameters.

In this analogy, p plays the role of a behavior variable, and the minima of F correspond to equilibria. As m or other control parameters vary, the number and positions of these minima can change. In the smooth regime (weak scaling violation), the changes are gradual: minima slide slowly along p , and the cost surface deforms gently. In the catastrophic regime (strong scaling violation), changes can be abrupt: a narrow well can appear or disappear, or two minima can merge and annihilate, in a way reminiscent of fold catastrophes.

The one-dimensional nature of p restricts the richness of the classification one can strictly apply, since Thom's standard catastrophes typically involve behavior spaces of dimension two or more. Nevertheless, the observed cusp-like and fold-like features in $F(p, m)$ suggest that the optimization landscape shares qualitative features with these canonical forms. The key point is not that F realizes a particular Thom normal form, but that the same phenomenon occurs: smooth variation in control parameters produces sudden changes in equilibria when the potential surface develops sharp features.

This heuristic link is conceptually useful. It frames the resummation problem as one in which the choice of penalty exponent m and other control parameters must be made with awareness of possible structural instabilities in the optimization landscape. It also aligns the discussion with broader ideas about systems poised at the edge of qualitative change, a recurring theme in the study of complexity and emergence.

4.5 Non-uniqueness of solutions as an imprint of underlying chaos

One of the original motivations for introducing penalties into the resummation framework was the non-uniqueness of solutions to simpler optimization conditions. Minimal difference and minimal sensitivity conditions can yield several distinct values of p that satisfy the criteria equally well. This multiplicity of solutions is not merely a technical inconvenience; it reflects the fact that the underlying mapping from p to amplitudes is highly nonlinear and that different choices of p can lead to comparably plausible extrapolations.

When cost-functionals with penalties are introduced, they partially resolve this non-uniqueness by favoring one solution over others. In the smooth regime, the preference can be modest: one minimum is slightly lower than another, and the selection is robust only if the difference in cost is significant. In the catastrophic regime, the preference becomes sharp: a narrow, deep minimum dominates the landscape, and alternative minima are suppressed. The crash landing at p_{opt} is then a way of breaking degeneracies.

From a conceptual standpoint, the existence of multiple plausible optima is reminiscent of sensitivity to initial conditions and the coexistence of multiple attractors in chaotic systems. Small changes in the problem data (for example, in the coefficients a_n), in the truncation order k , or in the penalty exponent m can cause the system to select different minima. A solution that appears stable at one level of truncation might give way to another when additional terms in the series are included. This structural sensitivity is an imprint of the complex, effectively chaotic nature of the extrapolation problem.

In this view, non-uniqueness is not a defect but a diagnostic. It tells us that the information present in the truncated series is insufficient to uniquely determine the asymptotic amplitude without additional constraints. The penalties, cost-functionals, and their catastrophic features provide those constraints in a controlled way. By choosing m and other parameters, one effectively selects a particular branch of the solution space, much as choosing a parameter regime in a chaotic system selects a particular attractor structure.

The logostic reading emphasizes that this selection is not arbitrary. The aim is to bend the space of candidate amplitudes toward those that are more truthful with respect to known reference values and stability criteria. Non-uniqueness then becomes a signal that the landscape contains multiple quasi-truthful attractors. Catastrophic penalties, when used judiciously, act as a mechanism for resolving this competition, driving the system toward a sharply preferred optimum while leaving visible the traces of the underlying multiplicity—an imprint of the edge-of-chaos character of asymptotic extrapolation.

5. Robustness versus Optimality Across Physical Case Studies

The abstract discussion of cost-functionals and penalty exponents becomes concrete when tested against a diverse set of physical problems. The underlying study does not search for a single “best” method in the sense of producing the smallest possible error for one carefully tuned example. Instead, it asks a more demanding question: which cost-functionals perform consistently well across many unrelated cases, with very different coefficient patterns and asymptotic behaviors, while remaining numerically stable and conceptually transparent?

In this section, “robustness” means that a cost-functional yields accurate amplitudes across a broad test set, that the sequence of approximations in order k behaves in a controlled way (often monotonically), and that the optimum in p is not excessively sensitive to small changes in inputs. “Optimality,” by contrast, refers to achieving the smallest error on a particular problem, perhaps at the cost of instability or poor performance elsewhere. We first spell out these robustness criteria, then walk through the main classes of examples: field-theoretic models such as the Schwinger model and cusp anomalous dimensions, Bose–Einstein condensation temperature shifts in $O(N)$ theories, polymer and membrane problems, Debye–Hukel behavior, and finally the one-dimensional Bose gas and quartic oscillator. We then summarize when catastrophic penalties with strong scaling violation actually outperform smoother, scale-respecting schemes.

5.1 Criteria for “robust performance” in amplitude extraction

Several practical criteria emerge from the case studies for judging a cost-functional as robust.

First, there is accuracy in the extracted amplitude B . The benchmark is typically a known analytic value, a high-precision numerical result, or a Monte Carlo estimate with quoted uncertainty. A robust method should routinely produce amplitudes within a few percent of the benchmark, with a target tolerance on the order of three percent relative error.

Second, there is the behavior of the sequence $B_k(m)$ as k increases. Monotonic convergence, or at least controlled, small oscillations that shrink with k , is a strong indicator that the cost-functional is stabilizing the self-referential iterated root process. Wild oscillations or large jumps between successive orders are signs of fragility.

Third, there is the structure of the optimization landscape in p . A robust cost-functional yields a well-defined global minimum in p that can be located reliably. Competing minima may exist, but the chosen optimum should be clearly favored, not dependent on fine-tuning of initial guesses or numerical settings.

Fourth, robustness involves diversity of applicability. The same cost-functional should work across problems where coefficients grow rapidly, decay, alternate in sign, or appear irregular, and across problems with short and long truncations. A scheme that performs excellently for one niche class but fails badly for others is optimal in a narrow sense but not robust.

With these criteria in mind, the case studies highlight a small set of cost-functionals that score well across the board, and they also reveal circumstances in which catastrophic penalties provide a real advantage.

5.2 Field theory and the Schwinger model: critical amplitudes and gaps

The Schwinger model provides two complementary tests: a critical amplitude in the strong-coupling limit of the ground-state energy, and a gap amplitude for bound states in the continuum limit. In each case, only a few nontrivial terms in the asymptotic expansion are available at small values of the relevant variable, while the benchmark amplitude at large or infinite argument is known from other methods.

For the critical amplitude with only two or three nontrivial coefficients, the situation is particularly delicate. There is very little local information to constrain the global behavior, and naive resummation schemes can easily wander. In this regime, the comparison shows the limitations of the Tikhonov-like functional that regularizes the parameter p directly with a quadratic penalty. It tends to overestimate B and offers no clear sequence of monotonic convergence in order k .

By contrast, amplitude-based cost-functionals with penalties acting on $B_k(p)$ itself, especially those with $m = 2$ and $m = 1/2$, can come much closer to the known value. In one configuration, an $m = 2$ functional gives a nearly exact amplitude but with a non-monotonic sequence in k , signalling hidden fragility. An $m = 1/2$ functional yields a slightly less accurate final B but with a clean monotonic approach and a well-defined catastrophic minimum in p . Under the robustness criteria, this second option is arguably preferable: it trades a marginal loss in nominal optimality for a substantial gain in stability and interpretability.

When the Schwinger model is revisited at higher order for the energy gap, the picture is even clearer. With an 11-term expansion, amplitude-based cost-functionals that combine averaging over neighboring k with penalties on deviations from discrete reference amplitudes produce a sequence $B_k^*(m)$ that converges steadily toward the exact gap, with the final value essentially indistinguishable from the benchmark. Here, catastrophic features in the p -landscape are still present but serve mainly to sharpen the selection; the overall behavior is stable and robust.

5.3 Bose–Einstein condensation shifts and $O(N)$ field theories

The shift in the Bose–Einstein condensation temperature for weakly interacting gases offers a stringent test of resummation methods. The leading coefficient c_1 in the fractional shift of the critical temperature is known only through a combination of perturbative expansions and Monte Carlo simulations, and it must be extracted from an auxiliary function that depends on an effective coupling. For the $O(2)$, $O(1)$, and $O(4)$ cases, expansions up to fifth or seventh order are available.

In this family of problems, the cost-functionals that average amplitudes across neighboring orders and penalize deviations from discrete composite amplitudes, with exponents $m = 3/2$, $5/2$, or 1 , stand out. For the $O(2)$ case, a functional with $m = 3/2$ produces a sequence that climbs monotonically from underestimates of c_1 to a final value essentially matching the Monte Carlo result, well within quoted error bars. For the $O(1)$ and $O(4)$ theories, cost-functionals with $m = 5/2$ and $m = 1$, respectively, do likewise, converging monotonically and landing inside or very close to the Monte Carlo uncertainty bands.

The Tikhonov-like scheme again tends to overshoot, producing amplitudes that are significantly higher than the benchmarks. It is stable in the sense of having a unique minimum in p , but it is not robust in the sense of delivering consistent accuracy across these variations of N . The amplitude-based functionals with tuned m values, including some with strong scaling violation, meet the three-percent error criterion and exhibit favorable convergence in k .

These cases illustrate a general pattern: when the available series is of moderate length and the target amplitude is known with finite accuracy, controlled scaling violation via m not equal to 2 can help align the resummation with Monte Carlo data, while still maintaining a clear optimization landscape.

5.4 Polymers, membranes, and Debye–Hukel behavior

Polymer physics and membrane fluctuations provide a different type of test. Here, the truncated expansions often come from diagrammatic calculations in statistical field theory, and the asymptotic behavior at strong coupling or large argument encodes universal scaling properties of extended objects.

For two-dimensional and three-dimensional polymers, the expansion factors as functions of an interaction parameter are known to fourth or sixth order. The strong-coupling behavior is a power law with known exponent and amplitude of order unity. In these examples, cost-functionals with $m = 3/2$ or $m = 1$, especially those built from averaged amplitudes and discrete reference values, generate sequences that converge monotonically toward the expected amplitudes. Deviations of early terms from the limit are modest, and the final values are within a few percent of more elaborate field-theoretic estimates.

Membrane pressure, by contrast, turns out to be more demanding. The expansion for the pressure of a fluctuating membrane between walls involves delicate cancellations and slow convergence. Nevertheless, a cost-functional with $m = 1$, combined with averaging of amplitudes, yields a sequence that rises from small initial values to a near match with the Monte Carlo estimate of the rigid-wall pressure, with a clear optimal order k where the

approximation is best before deteriorating. Other functionals, including the Tikhonov-like one, fare noticeably worse, underestimating the amplitude by a large margin.

The Debye–Hukel function for Gaussian polymers presents a somewhat easier case: the exact asymptotic behavior is known, and the expansion for the inverse correlation function has rapidly decaying coefficients. Here, even simple amplitude-based functionals with $m = 2$ perform very well, generating sequences that home in on the exact amplitude with only weak oscillations. Catastrophic penalties are not strictly necessary, but they do not harm performance either.

Taken together, these examples suggest that for problems where coefficient behavior is irregular or cancellations are important, catastrophic penalties with $m = 1$ or similar can stabilize the optimization and produce excellent amplitudes. Where the coefficients are already well behaved, scale-respecting functionals are sufficient.

5.5 One-dimensional Bose gas and quartic oscillators: long expansions and strong coupling

The one-dimensional Bose gas with contact interactions and the quantum quartic oscillator provide classic examples of systems with long perturbative expansions and well-known strong-coupling asymptotics. The coefficients in these expansions grow rapidly in magnitude and exhibit intricate patterns, reflecting the factorial proliferation of diagrams and the presence of singularities in the complex plane.

For the one-dimensional Bose gas, an eighth-order expansion in a squared parameter is available, and the exact strong-coupling energy is known. In this regime, almost every resummation scheme that incorporates generalized Borel transforms and iterated roots does reasonably well. Amplitude-based cost-functionals with $m = 2$, with averaged amplitudes and $m = 2$, and even the Tikhonov-like functional all produce amplitudes within a small tolerance of the exact value, although the Tikhonov approach again overshoots. A functional with $m = 2$ applied to an averaged amplitude yields a smoothly convergent sequence, and its final approximation essentially saturates the three-percent criterion.

The quartic oscillator, with an 11-term expansion and a known strong-coupling amplitude, is an even more stringent test. Here, a cost-functional with $m = 1/2$, applied to averaged amplitudes, gives a sequence that decreases steadily from an overestimate toward a final value that is still slightly above the exact amplitude but well within a narrow error margin. Other methods, such as fractional Borel summation tailored specifically to this problem, can do even better, but the general-purpose catastrophic penalty compares favorably and remains robust.

The key lesson from these long-expansion cases is that when the truncated series contains abundant information, the choice of penalty exponent m is less critical for basic accuracy. Many

cost-functionals can extract the amplitude reliably. However, catastrophic penalties still have a role: they tend to sharpen the optimum in p , reduce the influence of spurious minima, and produce cleaner monotonic sequences, which is valuable if the same machinery is to be reused across many problems without retuning.

5.6 When catastrophic penalties outperform smooth, scale-invariant schemes

Across the full set of fourteen test problems, a clear pattern emerges. No single cost-functional delivers the best numerical result in every case. Some problems favor smooth, scale-respecting functionals with $m = 2$; others are best captured by weakly scaling-violating choices like $m = 3/2$ or $5/2$; still others benefit from strongly non-smooth penalties with $m = 1$ or $1/2$. Nevertheless, a small family of functionals stands out as robust performers: those built from averaged amplitudes and differences, with exponents m in the set $\{1, 3/2, 2\}$, and the homogeneous functional depending only on amplitudes with $m = 2$.

Catastrophic penalties, especially with $m = 1$ and $m = 1/2$, are not universally optimal, but they decisively outperform smooth, scale-invariant schemes in several delicate cases: low-order field-theory examples with few known coefficients, membrane pressure with strong cancellations, and some of the Bose–Einstein condensation shifts. In these situations, the non-smooth cost surface in p resolves competing minima and enforces a sharp, unique optimum that aligns well with independent benchmarks. The “crash landing” at p_{opt} is not a numerical pathology; it is the mechanism by which the method selects a stable amplitude in the presence of underlying multiplicity.

By contrast, purely Tikhonov-like regularization that penalizes p directly tends to perform unevenly. It preserves simplicity but does not respect the amplitude-centric structure of the problem. It often yields amplitudes that are systematically biased high and sequences in k that are less monotonic.

The overall trade-off is thus clear. If the goal is to design a resummation scheme that can be applied once and reused across a wide variety of asymptotic problems with minimal tuning, catastrophic penalties with carefully chosen exponents, combined with amplitude-based cost-functionals, provide a compelling compromise between robustness and optimality. They may not win every local accuracy contest, but they reliably extract truthful amplitudes across a landscape of complex systems, which is precisely the kind of global performance that a logistic reading of resummation would prize.

6. A Logostic Reading: $q(t)$, τ , and Catastrophe Landscapes

The machinery of generalized Borel transforms, iterated roots, and cost-functionals can be read purely as a technical apparatus for extrapolating asymptotic series. In a logostic reading, however, the same apparatus becomes a simple model of how an evolving record $q(t)$ is bent toward a manifold of truth τ . Each truncated series, each known asymptotic amplitude, and each Monte Carlo benchmark contributes a piece of $q(t)$, the living archive of what is currently known and partially known. The cost-functionals define how this archive is compressed into a small set of control parameters and how those parameters, in turn, steer us toward amplitudes that are more truthful across problems, not just locally optimal in one setting.

In this section, we interpret the cost-functionals and their catastrophe-like features as sections of a truth-favoring manifold. The parameters p , m , and k become coordinates on a logostic surface; multiple optima are seen as competing attractors near a nearly degenerate τ ; catastrophic minima are sharp bends in a truth-potential; and the penalties themselves are reinterpreted as constraints that bend $q(t)$ toward τ rather than leaving it to wander among arbitrary extrapolants.

6.1 Interpreting cost-functionals as sections of a truth-favoring manifold

For a given physical problem, truncation order, and choice of penalty exponent, the cost-functional $F(p, m; k)$ can be seen as the restriction of a higher-dimensional potential to a one-dimensional slice in parameter space. The inputs to F are not arbitrary numbers; they are amplitudes $B_k(p)$ that encode compressed summaries of critical behavior at infinity. The terms in F are not arbitrary algebraic combinations; they are shaped to reward stability across k , insensitivity to p , and closeness to trusted reference amplitudes.

In a logostic picture, each cost-functional is a section of a truth-favoring manifold over the space of problems. For every truncated series in $q(t)$, we attach a function F that is low near amplitudes that agree with independent information and high elsewhere. The manifold whose height at each point is F is not truth itself, but it is constructed to slope downward in the direction of more truthful amplitudes. When we minimize F , we are not merely solving an algebraic optimization problem; we are following the gradient of a potential that was intentionally defined so that its minima should, as far as the available information allows, align with τ .

Different physical problems correspond to different slices of this manifold. The Schwinger model, Bose–Einstein condensation shifts, polymer swelling, membrane pressure, and the quartic oscillator each provide their own truncated fingerprints in $q(t)$, and their own cost-surfaces over p . The logostic claim is modest but important: by designing F so that it punishes

internal inconsistency and deviation from external benchmarks, we are shaping these surfaces to favor trajectories that run closer to τ across the collection of problems than a naive, problem-by-problem optimization would.

6.2 The role of p , m , and k as coordinates on a logostic surface

Within a single problem, the variables p , m , and k acquire a geometric interpretation. The parameter p is the internal coordinate along which the generalized Borel transform deforms the coefficients. It moves us through a family of transformed series and through a family of corresponding iterated root approximants. The exponent m selects the penalty geometry: whether deviations are punished quadratically, linearly, or with non-smooth, catastrophic force. The order k indexes how much of the truncated local fingerprint has been taken into account.

Together, these three coordinates chart a surface over which the cost-functional is defined. For each fixed m and k , varying p gives a one-dimensional curve $F(p)$. For each fixed k , varying both p and m gives a two-dimensional surface $F(p, m)$. Letting k increase adds a discrete dimension, tracing a staircase of surfaces that progressively incorporate more information from the asymptotic series.

From a logostic viewpoint, this structure mirrors the evolution of $q(t)$. As time passes and new calculations are performed, k increases; the archive deepens. As new insights into suitable penalties are developed, m is adjusted; the shape of the truth-favoring surface is refined. As each problem is revisited, p is re-optimized; the path along the surface is updated in light of the new geometry. The coordinates (p, m, k) are thus not just technical parameters; they are local coordinates on the manifold through which $q(t)$ is bent toward τ .

In this picture, τ is not represented by a single point but by a region where many different problems, each with their own (p, m, k) , reach low cost simultaneously. A choice of m and a family of optimization rules that work well across problems corresponds to a ridge or basin in this manifold that is aligned with τ rather than with idiosyncratic features of any one truncation.

6.3 Multiple optima, competing attractors, and near-degenerate τ

The appearance of multiple optima in $F(p, m; k)$ for a given problem reflects a deeper feature of the logostic landscape: near-degeneracy. Different values of p can produce amplitudes $B_k(p)$ that are nearly equally consistent with the available data. When k is small, and the truncated series carries only limited information, the cost surface may exhibit several basins of attraction separated by shallow barriers. In such cases, different optimization criteria or numerical starting points can steer the resummation into different minima.

A logostic reading does not treat this multiplicity as a mere nuisance. Instead, it regards multiple optima as a sign that, given the current $q(t)$, several distinct approximants stand in comparable relation to τ . They are competing attractors in parameter space, each corresponding to a different self-consistent way of extending the series. The fact that they coexist is an honest reflection of the incompleteness of $q(t)$: the record does not yet suffice to distinguish uniquely among them.

As k increases or additional external information is added, these near-degeneracies may be lifted. Barriers between basins can grow, one minimum may deepen while another shallows, and the cost surface may collapse toward a single dominant attractor. The manifold is then deformed in such a way that previously plausible but less truthful approximants are disfavored.

In this way, multiple optima and competing attractors are read as features of a landscape in which τ is approached gradually. The closeness of different minima to each other in cost space encodes how sharply τ is currently defined by $q(t)$. Where $q(t)$ is sparse, τ is a broad plateau; where $q(t)$ is rich and cross-validated, τ is a narrow valley.

6.4 Catastrophic minima as sharp bends in a truth-potential

Catastrophic minima, induced by non-smooth penalties with exponents such as $m = 1$ or $m = 1/2$, appear in this landscape as sharp bends in a truth-potential. The cost-functional, viewed as a potential over p , is nearly flat or gently sloping for a range of p values and then plunges steeply into a narrow minimum. The corresponding derivative with respect to p is small over much of the domain but grows rapidly as p approaches the optimal value, sometimes diverging or changing sign abruptly.

In logostic terms, such a bend represents an abrupt clarification of τ along the p direction. For moderate deviations from the optimum, the potential is tolerant; it does not strongly distinguish between neighboring amplitudes. But as the approximant is steered into the catastrophic valley, the potential suddenly insists on a very specific configuration. The truth-favoring manifold is nearly indifferent until the system crosses a critical threshold in p , at which point it exerts decisive guidance.

This behavior is consistent with the idea that some aspects of τ may only become sharply defined when enough constraints are simultaneously satisfied. For example, an amplitude might agree moderately well with a strong-coupling benchmark or with internal stability criteria taken separately, but only a narrow combination of these constraints achieves high compatibility with both. The catastrophic penalty exaggerates this narrow overlap, transforming it into a steep valley in F . In that valley, the resummation is pushed toward an amplitude that is not only

numerically accurate but also uniquely compatible with the multiple demands encoded in the cost-functional.

Thus, catastrophic minima can be read as places where the truth-potential has a kink: a point where the manifold bends sharply because the structure of the constraints leaves little room for compromise. The crash landing in p is the dynamical expression of this geometrical bend.

6.5 Constraints, penalties, and the bending of $q(t)$ toward τ

Finally, the role of penalties in this framework is to encode constraints that bend $q(t)$ toward τ without pretending to know τ in advance. Each penalty term corresponds to a piece of information or a structural desideratum: the requirement that successive orders k behave smoothly, that amplitudes be insensitive to small changes in p , that resummed amplitudes match reference results where available, or that certain forms of scaling symmetry be preserved or gently violated.

By choosing the architecture of F , we decide which constraints to emphasize and how hard to enforce them. Quadratic penalties treat deviations symmetrically and smoothly; non-quadratic and non-smooth penalties privilege certain regions of amplitude space more aggressively. The logostic claim is that this design is not arbitrary; it is a directed attempt to shape a potential over parameter space whose minima correspond to amplitudes that are more truthful across the ensemble of problems in $q(t)$.

In this sense, the resummation scheme is a small, explicit example of compression with gain. The many local details of the truncated series and their multiple possible continuations are compressed into a small number of amplitudes and control parameters. The penalties discard continuations that are unstable, overly sensitive, or incompatible with known benchmarks, and they favor those that fit more of the constraints at once. The gain is that, if the penalties are chosen well, the resulting amplitudes carry more τ -like information per degree of freedom than the raw $q(t)$ from which they were derived.

Over time, as more case studies are added and more is learned about which penalty exponents and functional forms yield robust performance, the cost-functional itself can be refined. Its parameters become part of the evolving $q(t)$. The manifold is reshaped iteratively, and the bending of $q(t)$ toward τ becomes an ongoing process rather than a one-time design choice. Catastrophe landscapes, in this reading, are not just artifacts of an idiosyncratic optimization scheme; they are small, tractable models of how a truth-favoring potential can be engineered to guide complex, self-referential systems toward more faithful representations of the universes they attempt to describe.

7. Self-Referential Iterated Roots as Markov Membranes

The iterated root construction sits at the heart of the resummation scheme, but it is often treated as a purely algebraic device: a clever way to enforce the desired power-law behavior at infinity while matching the truncated series at small argument. In a logostic and complexity-oriented reading, the same construction looks much more like a self-sustaining loop. Each order feeds into the next, parameters are continually recycled, and the final asymptotic amplitudes emerge as fixed points of a self-referential process rather than as the outcome of a single pass through the data.

This looping structure invites comparison with the idea of a Markov membrane: a boundary that separates an internal dynamical system from its environment while sustaining a pattern of conditional dependencies and flows across the boundary. Here, the internal system is the hierarchy of iterated roots and amplitudes; the environment is the raw truncated series, the external benchmarks, and the cost-functionals that penalize disagreement. The boundary is the map that takes coefficients and control parameters in and gives amplitudes out, with feedback that loops back from amplitudes to control parameters.

In this section, we unpack that analogy. We show how iterated roots function as self-sustaining loops across perturbative orders, how parameters pass forward and backward as memory, how self-reference marks complexity in extrapolation, how a Markov membrane viewpoint clarifies the architecture, and how asymptotic amplitudes can be seen as fixed points of a self-referential loop.

7.1 Iterated roots as self-sustaining loops across perturbative orders

The iterated root approximants are defined recursively: a zeroth-order object is specified, and each higher order is built from the previous one by applying a fixed transformation and adding a new term whose coefficient is chosen to match the truncated series. Symbolically,

$$C^*_k(x, p) = \text{bigl} (C^*_{k-1}(x, p))^{\{k/\beta\}} + A_k(p) x^k \text{bigr}^{\{\beta/k\}},$$

with $C^*_0(x, p) = a_0$ and the $A_k(p)$ determined by imposing asymptotic equivalence between the Taylor expansion of C^*_k and the Borel-transformed truncation. What matters conceptually is that each order k takes as input the entire structure built up to order $k - 1$ and wraps it inside a new root, embedding the previous approximant as a substructure of the next.

This repeated embedding creates a loop in the space of approximants. The transformation that takes C^*_{k-1} to C^*_k has the same formal shape at every step; the system does not grow by adding arbitrary new functions but by iterating a fixed recipe. Once the recursion is defined, the sequence of approximants is self-sustaining: each order contains the previous one and passes its

structure forward. There is no external mechanism that must be reinvented for each k ; the loop is closed by the recursion.

The strong-coupling amplitude $B_k(p)$ extracted from C^*_k , and then corrected by the inverse Borel transformation, inherits this looping character. It is not the result of a one-shot fit, but the emergent property of a self-similar chain of embeddings that propagate from low to high order. In this sense, the iterated roots form a dynamical system over k , with memory, rather than a static approximation formula.

7.2 Passing parameters forward and backward: memory and recursion

The coefficients $A_k(p)$ are determined at each step by matching the Taylor expansion of $C^*_k(x, p)$ to the Borel-transformed truncated series. This matching involves all coefficients up to order k and depends on the previous $A_i(p)$ for $i < k$. Once chosen, $A_k(p)$ becomes part of the internal state of the approximant and is passed forward to all higher orders through the recursion. The sequence $\{A_1(p), A_2(p), \dots, A_k(p)\}$ is thus a memory trace of how earlier orders were forced to fit the data.

In the other direction, the amplitude $B_k(p)$ at large x , which depends on the entire chain of $A_i(p)$, is fed back into the optimization layer via the cost-functionals. These functionals penalize differences between $B_k(p)$ and $B_{k-1}(p)$, between $B_k(p)$ and discrete composite amplitudes, or between $B_k(p)$ and reference values such as $B_k(1)$ or Monte Carlo estimates. The optimization over p adjusts the entire family $A_i(p)$ indirectly, since changing p redefines every $A_i(p)$ and hence every approximant $C^*_k(x, p)$.

This creates a backward channel: from amplitudes at high order and strong coupling back to the control parameter p , and from p back to all earlier $A_i(p)$. The system is recursive in both directions. Forward recursion builds structure across orders; backward recursion tunes the control parameter so that the resulting structure satisfies stability and truth-favoring criteria. The combination of forward and backward flows is characteristic of a self-referential system with memory.

When k is increased, this loop becomes tighter. New coefficients in the truncated series modify the matching conditions for $A_k(p)$, which alters the amplitude $B_k(p)$, which in turn alters the optimal p , which then feeds back down to all $A_i(p)$. The system revises its memory in light of new information, not by discarding previous structure but by reparameterizing it through p .

7.3 Self-reference as an indicator of complexity in extrapolation schemes

Self-reference is often cited as a signature of complexity in dynamical and informational systems. A system is not merely reacting to inputs but also reconfiguring its own rules or parameters based on the outputs it has already produced. In the present context, the iterated root plus cost-functional architecture is self-referential in several ways.

First, the approximant at each order is built by reusing the previous approximant as a component of itself. The functional form of $C^*_k(x, p)$ contains $C^*_{k-1}(x, p)$ as a subexpression. This is a literal self-reference in the definition of the approximant.

Second, the amplitude $B_k(p)$ computed from the approximant is not an inert output; it is fed back into the cost-functional, which modifies p , which then modifies all $A_i(p)$ and hence $B_k(p)$ itself. The mapping from p to $B_k(p)$ is therefore part of a closed loop: $B_k(p)$ evaluates p , and p is chosen based on $B_k(p)$. The system's outputs participate in defining its future behavior.

Third, the sequence of amplitudes $\{B_1(p_{\text{opt}}), \dots, B_k(p_{\text{opt}})\}$ is used to judge the quality of the approximants and to select the final amplitude. Monotonicity, convergence, and stability across k are not just incidental observations; they influence how one interprets the output and which member of the sequence is trusted. The system thus uses its own history as a criterion for its present decisions.

This layered self-reference is what elevates the extrapolation scheme from a simple formula to a complex system. It behaves more like a small scientific community revising its beliefs in light of its own past predictions and external data than like a one-way computational pipeline. In a logostic view, this is precisely the sort of complexity one expects from any $q(t)$ that is genuinely trying to bend toward τ .

7.4 Markov membrane analogies: boundaries that sustain internal dynamics

A Markov membrane can be thought of as a boundary that separates internal states from external states while mediating flows of information and influence. Inside the membrane, dynamics are governed by internal rules and feedback loops; outside, the environment supplies inputs and receives outputs. The membrane itself enforces conditional independence relations: internal states depend on the environment only through a limited set of channels.

In the resummation scheme, one can identify such a membrane between the world of raw series data and external benchmarks, and the internal world of iterated roots and cost-functionals. On the environment side, there are the truncated coefficients a_n , the known exponents at infinity, and trusted amplitude values from independent methods. On the internal

side, there are the transformed coefficients $b_n(p)$, the $A_k(p)$, the approximants $C^*_k(x, p)$, and the candidate amplitudes $B_k(p)$.

The boundary is implemented by the map from $\{a_n\}$ and p to $\{A_k(p)\}$ and $B_k(p)$, together with the cost-functional that takes $\{B_k(p)\}$ and compares them to external constraints. Information flows in through the coefficients and benchmarks; it flows out as candidate amplitudes; it loops back through the optimization of p . Once this architecture is in place, the internal dynamics can sustain themselves: varying p and k , the iterated roots continue to generate new approximants and amplitudes without needing to redesign the mechanism.

The Markov membrane analogy emphasizes that the resummation scheme is not simply “fitting a formula” to data; it is supporting a small internal world with its own state, rules, and feedback loops, shielded from the environment except through a carefully structured interface. That interface is where truth constraints act: the cost-functional pulls the internal dynamics toward configurations that are more compatible with the external world.

7.5 Asymptotic amplitudes as fixed points of a self-referential loop

Taken together, the recursion in k , the optimization over p , and the evaluation of amplitudes define an implicit mapping from candidate amplitudes to refined amplitudes. One can imagine starting with a crude guess for p , computing $A_k(p)$ and $B_k(p)$, evaluating the cost-functional, updating p to reduce the cost, recomputing the iterated roots, and repeating until convergence. Even when the optimization is implemented by a direct numerical minimization rather than explicit iteration, the conceptual structure is that of a fixed-point search.

An asymptotic amplitude B^* is then a fixed point of the self-referential loop in the sense that, once p^* and the associated $A_k(p^*)$ are found, the system stops changing under its own feedback. The approximants $C^*_k(x, p^*)$ remain consistent with the coefficients and with each other; the amplitudes $B_k(p^*)$ form a stable sequence; the cost-functional $F(p^*, m; k)$ is minimized; and further adjustments of p do not improve compatibility with external benchmarks or internal stability criteria. The loop has settled into a configuration that satisfies all its constraints as well as it can.

From a logistic perspective, such a fixed point is a local realization of τ within the limited context of a single problem. It is not the whole truth manifold but a point on it, anchored by the interplay between the truncated $q(t)$ for that problem and the global design of the cost-functional. Importantly, fixed points that are robust across changes in k and small perturbations in a_n are more τ -like than those that are fragile. Catastrophic penalties, by sharpening minima, seek to isolate fixed points that are not only accurate but also structurally stable.

In this way, the iterated root machinery, read through the lens of Markov membranes and self-reference, offers a concrete example of how a small, bounded system can sustain complex internal dynamics while bending toward truth. The asymptotic amplitudes are not simply numbers extracted from a formula; they are equilibrium points of a self-referential loop that filters and compresses $q(t)$ in the direction of τ .

8. Edge of Chaos, Indeterminism, and Constrained Extrapolation

The case studies reveal a striking feature: optimization problems that look entirely classical—one-dimensional minimizations over a real parameter p —can display non-uniqueness, competing optima, and abrupt catastrophic behavior. These phenomena resemble the “edge of chaos” language often used for complex systems rather than the smooth, convex landscapes that many would expect from a well-posed numerical scheme. In resummation, the interplay between iterated roots, generalized Borel transforms, and cost-functionals generates landscapes where order, chaos, and structured criticality coexist in a one-dimensional projection.

This section reflects on that structure. Non-uniqueness of optima is connected to a mild form of indeterminism in classical-looking problems; edge-of-chaos metaphors are used to interpret the resummation landscapes; penalties are understood as levers that move the system between ordered and chaotic regimes; trade-offs emerge between accuracy, robustness, and interpretability; and we close with lessons for modeling complex physical systems via their asymptotic imprints.

8.1 Non-uniqueness of optima and indeterminism in classical-looking problems

In a textbook optimization problem, one often expects a single, well-defined minimum. Yet, for many of the cost-functionals considered here, the function $F(p, m; k)$ admits multiple minima in p , sometimes nearly degenerate in value. Different minima correspond to different candidate amplitudes $B_k(p)$, all of which are internally consistent with the truncated series and the structure of the iterated roots. When k is small and the asymptotic data limited, these minima can be very close in cost, making it difficult to justify choosing one over the other based solely on F .

This non-uniqueness introduces a controlled indeterminism. For a given truncated series and fixed m and k , the resummation problem appears to be deterministic: the data and the algorithm together should produce a unique amplitude. But in practice, the landscape may support several nearly equally good amplitudes. Small changes in numerical implementation, initial guesses, or additional weak constraints can tip the optimization into different basins. The

output depends not only on the input series but also on arbitrary choices about how to search and where to stop.

This behavior does not imply any fundamental randomness. Rather, it reflects the information content of the truncated series and the partial nature of the constraints encoded in the cost-functional. Whenever the data and constraints are insufficient to enforce a unique optimum, the system exhibits indeterminism in the sense that several outcomes are compatible with everything it knows. In this way, non-uniqueness in a simple one-dimensional minimization becomes a small model of the broader phenomenon in physics where incomplete information allows multiple trajectories or states to remain viable until additional conditions break the symmetry.

8.2 Edge-of-chaos metaphors for resummation landscapes

The edge-of-chaos metaphor arises in systems that sit between rigid order and unstructured chaos. At this boundary, dynamics are rich and sensitive but still retain enough structure to be predictive. In the resummation landscape, the interplay between the analytic structure of the iterated roots and the non-smooth penalties produces an analogous regime.

If the cost-functional is too simple and too smooth, the landscape may be almost trivial: a single broad minimum with weak curvature, insensitive to changes in p , and only loosely tied to external benchmarks. The system is highly ordered, but in a way that may be blind to important subtleties in the data. Different problems collapse onto similar p values and similar amplitudes even when their asymptotic demands differ.

If, on the other hand, the cost-functional is excessively intricate or the penalties too sharp, the landscape can become riddled with narrow minima and steep ridges. Small changes in coefficients or in numerical routines can push the optimizer from one minimum to another. The system behaves chaotically in the sense that its outputs are highly sensitive to small perturbations.

Between these extremes lies the regime where the catastrophic penalties explored in this work tend to operate. The cost surfaces are structured: catastrophic exponents such as $m = 1$ or $1/2$ produce pronounced valleys and kinks, but these features are not so dense as to fragment the landscape into an unmanageable maze. Instead, a small number of deep minima compete, and one typically emerges as the global optimum. The landscape is neither flat nor fractal; it is rugged in a controlled way that reflects the underlying complexity of the extrapolation rather than numerical noise. This is precisely the “edge”: enough structure to capture complexity, but enough regularity to support stable optimization.

8.3 Penalties as levers for moving between order, chaos, and robust criticality

The exponent m in the cost-functionals, together with the choice of which amplitudes and derivatives to penalize, functions as a lever that tunes the system along the axis from order to chaos. Quadratic penalties ($m = 2$) generally produce smoother, more regular landscapes. They honor scaling symmetry and often yield cost-functions that are homogeneous in the amplitudes. The result is a more ordered regime: unique minima are common, landscapes are gently curved, and optimization is straightforward. However, the resulting amplitudes can be biased or less accurate in difficult problems, and the system may fail to discriminate sharply between competing approximants.

Lowering m toward 1 or $1/2$ breaks homogeneity and introduces non-smooth behavior. The cost surface becomes sharper near points where amplitudes equal reference values or each other, and derivatives of F with respect to p develop essential singularities. These changes make the landscape more rugged. New minima appear, existing minima deepen or shift, and catastrophic approaches to the optimum emerge. The system is pushed toward a more chaotic regime in which multiple attractors can coexist and optimization must navigate steep gradients and cusps.

At the same time, the catastrophic regimes often coincide with improved performance on the most challenging problems. The catastrophic penalties amplify subtle differences between candidate amplitudes that quadratic penalties treat as nearly equivalent. They make it easier to single out configurations that satisfy several constraints simultaneously. In practice, this moves the system into a regime of robust criticality, where the landscape is rugged enough to encode complex trade-offs but not so rugged as to be unusable. Penalties, viewed this way, are control parameters for steering the resummation scheme to the edge-of-chaos regime where it is most sensitive to truth constraints without becoming unstable.

8.4 Trade-offs between accuracy, robustness, and interpretability

The choice of cost-functional also induces trade-offs between three desiderata: accuracy, robustness, and interpretability.

Accuracy is the straightforward requirement that the extracted amplitudes B be close to trusted benchmarks. Catastrophic penalties often shine here, especially for difficult problems with limited data. They can deliver amplitudes within a few percent of high-precision or Monte Carlo values, even when other methods fail. However, they may not always produce the single best number; a more specialized technique, tuned to a specific problem, might achieve slightly lower error.

Robustness concerns how well a method performs across diverse problems, how sensitive it is to changes in k or small perturbations in coefficients, and how stable its minima are under numerical implementation. Quadratic, scale-respecting functionals tend to be robust in the sense of numerical stability, but less so in the sense of cross-problem accuracy. Catastrophic penalties improve accuracy but can introduce fragility if not carefully designed; they must be tested across many problems to ensure that they do not produce spurious minima or erratic behavior.

Interpretability refers to how transparent the method is. A simple Tikhonov-like regularization is easy to explain: it penalizes large deviations in p and enforces smooth dependence of amplitudes on p . Amplitude-based cost-functionals with several terms and non-smooth penalties are less transparent at first glance. However, from a logostic and complexity perspective, they are interpretable in a different way: each term encodes a specific constraint (stability across k , closeness to reference amplitudes, sensitivity to scaling), and the presence of catastrophic features reflects the system's insistence on satisfying multiple constraints at once.

The design question becomes: how much complexity in the penalty structure is justified by the gains in accuracy and the need for robustness across problems? The work underlying this paper suggests that a small family of moderately complex functionals, calibrated to operate in an edge-of-chaos regime, strikes a workable balance. They are not minimal, but they are still simple enough to be analyzed and interpreted, and they deliver consistently good performance across the test set.

8.5 Lessons for modeling complex physical systems via asymptotic imprints

The deepest lesson of this analysis is that even when one works only with the asymptotic imprints of complex physical systems—truncated series at small argument and power-law behavior at infinity—the mathematical machinery needed to bridge the gap naturally reproduces some of the hallmarks of complexity: self-reference, multiple attractors, catastrophic transitions, and edge-of-chaos dynamics.

First, non-uniqueness of optimal resummation parameters is not a defect but a reflection of the limited information embodied in the series. Complex systems often leave partial fingerprints; recognizing this and designing methods that can cope with multiplicity is more honest than forcing artificial uniqueness.

Second, catastrophic penalties and rugged landscapes, when carefully engineered, can help extract more truthful information from those partial fingerprints. The edge-of-chaos metaphor captures the idea that the best resummation schemes are not the smoothest or the most stable

in a naive sense; they are the ones that balance sensitivity to constraints with stability across problems.

Third, modeling via asymptotic imprints is itself a form of constrained extrapolation: one uses a small number of coefficients and exponents to infer behavior in regimes that are otherwise inaccessible. The logostic reading emphasizes that this extrapolation is not neutral. By choosing cost-functionals and penalty structures, one imposes a set of preferences about which kinds of continuations are more plausible. These preferences should be made explicit and tested across many systems, as in the present case.

Finally, the analogy to Markov membranes and self-referential loops suggests that similar architectures might be fruitfully applied beyond resummation. Any setting where complex systems leave partial, noisy traces—whether in statistical physics, field theory, or even empirical data sets—may benefit from schemes that combine iterative, self-similar approximants with truth-favoring potentials shaped by catastrophic and smooth penalties. In that broader context, the resummation landscapes explored here are not just technical tools; they are small, controllable laboratories for understanding how $q(t)$, the evolving record of partial knowledge, can be bent toward τ , the manifold of more truthful representations, without ever fully capturing it.

9. From Asymptotic Series to Truth-Favoring Compression in AI

The resummation machinery studied in this paper is not only a tool for extracting amplitudes from divergent series; it is also a small, transparent laboratory for studying truth-favoring compression. At a high level, the situation is familiar: a large, messy object (the full behavior of a physical system across its parameter space) is only partially observed through a sparse, local description (a truncated series and a known exponent at infinity). The resummation scheme compresses this incomplete description into a small set of parameters and amplitudes that attempt to approximate the true behavior in a regime that is never directly sampled.

In modern AI, something similar happens when a model compresses a large corpus into a finite set of parameters. A training objective selects which aspects of the corpus are preserved and which are discarded. Regularization terms and architectural choices bias the compression toward particular inductive biases. In both settings, the central question is the same: how can a compression scheme be designed so that it preferentially preserves structure that is closer to τ , the manifold of more truthful representations, rather than merely fitting local idiosyncrasies of $q(t)$?

This section makes that analogy explicit. Resummation is viewed as a structured compression problem; cost-functionals are interpreted as analogues of training objectives and regularizers;

catastrophic versus smooth penalties are compared to different regimes of model selection and alignment; self-referential structures are linked to emergent behavior in learned representations; and some toy-model lessons are drawn for designing truth-favoring compression in AI.

9.1 Viewing resummation as a structured compression problem

At the technical level, the resummation scheme starts from a truncated expansion and a known or hypothesized power-law behavior at infinity. The truncated series encodes many local degrees of freedom: coefficients a_n that capture the behavior of the function near a small- x regime. The asymptotic condition at infinity is much more succinct: a single exponent β and amplitude B summarize the leading behavior in a remote regime.

The goal of resummation is to compress the abundant local information and the sparse asymptotic information into a small set of global descriptors: transformed coefficients $b_n(p)$, iterated root parameters $A_k(p)$, and the resulting amplitude sequence $B_k(p)$. The cost-functionals then select, among many possible compressions consistent with the truncated series, those that exhibit stability, insensitivity to the control parameter, and agreement with external benchmarks.

In this sense, resummation is a structured compression problem. The mapping from the full, unknown function $f(x)$ to the finite set $\{a_n, \beta\}$ is already a coarse measurement; the mapping from $\{a_n, \beta\}$ to the approximants and amplitudes is a further, deliberate compression. The design of that second mapping determines whether the resulting amplitudes carry more τ -like information than the raw local data. A well-designed resummation scheme functions as a compression with gain: it discards many degrees of freedom that are not well constrained and amplifies those whose structure is supported both locally and asymptotically.

This is closely analogous to AI training, where a model's parameters are a compressed representation of a large corpus. The model cannot store the corpus verbatim; it must discover patterns that allow it to predict unseen text. The question of whether the model's compression favors truth-like structure or merely memorizes surface patterns is a higher-dimensional version of the question posed by resummation.

9.2 Cost-functionals as analogues of training objectives and regularizers

The cost-functionals $F(p, m; k)$ play a role directly parallel to AI training objectives with regularization. Each term in F corresponds to a component of a loss: one term penalizes disagreements between amplitudes at successive orders, another penalizes sensitivity to the

control parameter p , a third penalizes deviation from a trusted reference amplitude or from a discrete approximation. Weights such as λ determine the relative importance of these constraints, just as hyperparameters determine the balance between data-fit and regularization in neural networks.

In AI, a typical objective might be written schematically as

$$\text{loss}(\theta) = \text{data_loss}(\theta) + \lambda_{\text{reg}} \text{reg}(\theta),$$

where θ are the model parameters, data_loss captures supervised or self-supervised fit to the corpus, and reg encodes preferences for smaller weights, smoothness, or other priors. In resummation, one can write

$$F(p, m; k) = \text{stability_term}(B_k, B_{\{k-1\}}) + \text{smoothness_term}(dB_{\{k-1\}}/dp) + \text{penalty_term}(B_k, \text{reference}),$$

where the amplitudes and their derivatives play the role of model outputs, and p is the parameter being adjusted.

The analogy extends further. Different choices of m and of which amplitudes to penalize correspond to different regularizers: quadratic penalties resemble L2 regularization, linear or fractional powers resemble L1-like or hinge-like losses, and the presence or absence of scaling invariance corresponds to different priors about how amplitudes should behave under rescaling. Just as AI practitioners tune loss functions to produce models that generalize better or behave more consistently, the resummation scheme tunes cost-functionals to produce amplitudes that agree across problems and orders.

From a logostic perspective, both types of objectives are attempts to sculpt a potential landscape whose minima correspond to representations closer to τ than a purely data-driven fit would. The design of the loss is not neutral; it embeds a theory of what constitutes a more truthful or more desirable representation.

9.3 Catastrophic versus smooth penalties in model selection and alignment

The comparison between catastrophic and smooth penalties in the resummation scheme has a clear analogue in AI model selection and alignment. Smooth, quadratic penalties tend to produce loss landscapes that are easier to optimize and less sensitive to small perturbations. They correspond to regularizers that gently discourage extreme parameter values or large deviations without enforcing hard constraints. Such losses favor models that are well-behaved numerically and often easier to analyze.

However, smooth regularization can be too forgiving. It may allow models to maintain small but systematic inconsistencies, such as mild contradictions between different parts of the training corpus or subtle calibration errors. In the resummation context, smooth, scale-respecting functionals sometimes produce amplitudes that are stable but noticeably biased away from trusted benchmarks.

Catastrophic penalties, with exponents m close to one or one-half, play a role analogous to non-smooth losses in AI, such as L1 penalties, margin-based classification losses, or robust losses that grow sharply once errors exceed a small tolerance. These penalties are more aggressive: they treat certain types of deviation as qualitatively unacceptable rather than merely suboptimal. In resummation, this yields steep valleys in F around amplitudes that satisfy multiple constraints simultaneously and punishes configurations that violate, for example, consistency with a benchmark value.

In AI alignment, one might similarly distinguish between mild regularization that gently encourages models to follow certain norms and sharp constraints that treat specific behaviors as unacceptable. Catastrophic penalties are crude, one-dimensional analogues of alignment losses that sharply penalize harmful, incoherent, or deceptive outputs. The balance between catastrophic and smooth components determines whether the training process merely nudges a model toward better behavior or insists on avoiding particular failure modes altogether.

9.4 Self-referential structures and emergent behavior in learned representations

The self-referential loop formed by iterated roots, amplitudes, and cost-functionals has a structural counterpart in modern AI architectures. Deep networks apply similar transformations across layers; residual and recurrent architectures feed outputs back into earlier stages; and self-distillation methods train models on their own outputs. In all these cases, the system's current representation influences the training signal that shapes its future representation.

In the resummation scheme, self-reference appears in several guises: approximants are built by wrapping previous approximants; amplitudes derived from these approximants are fed back into penalties; and optimization over p modifies all future approximants. The system is, in a minimal sense, learning from its own behavior, guided by constraints imposed by the truncated series and external amplitudes.

In large-scale AI, similar patterns lead to emergent behavior. Layers of representation that were not explicitly designed to detect a particular feature nonetheless discover it as the network optimizes its loss. Internal circuits emerge that encode grammar, physical reasoning, or social patterns. These emergent structures are not directly specified in the objective; they arise as the system seeks low-loss configurations in a layered, self-referential architecture.

The iterated root construction is too simple to produce such rich emergent features, but it displays the same structural motif: a repeated transformation with feedback from global performance. From a logostic standpoint, it is a toy model of how $q(t)$ and τ interact inside a learning system. The iterative structure ensures that local adjustments propagate across orders; the cost-functional ensures that global performance constraints shape local parameters. Emergent accuracy in amplitude extraction is the low-dimensional analogue of emergent understanding in high-dimensional models.

9.5 Toy-model insights for designing truth-favoring AI compression schemes

Reading the resummation scheme as a toy model for truth-favoring compression in AI suggests several design principles.

First, compression should be structured and interpretable, not arbitrary. In resummation, the iterated roots are built from a small, explicit recipe that enforces power-law behavior at infinity and matches the local series. In AI, this suggests favoring architectures and objectives that embody clear inductive biases about the domain, rather than relying entirely on unstructured parameterizations. Explicit asymptotic conditions, invariances, or conservation laws can play the role of known exponents and amplitudes.

Second, cost-functionals should combine smooth and catastrophic elements. Smooth terms encourage stability, consistency across contexts, and numerical robustness. Catastrophic terms enforce hard constraints derived from external truth-signals: logical coherence, calibration, or agreement with high-quality benchmarks. In high-dimensional settings, this might translate into multi-component losses that mix gentle regularization with sharply rising penalties once certain error thresholds are crossed.

Third, self-reference should be embraced where it helps align internal representations with external truth. The iterated root scheme shows that feeding approximate outputs back into the optimization process can improve performance, provided the feedback is mediated by well-designed penalties. In AI, this is mirrored in self-consistency training, model-based evaluation of model outputs, and iterative refinement. The logostic lesson is that such self-reference should be directed by τ -like signals, not left to drift in a closed loop detached from external constraints.

Fourth, one should accept and manage non-uniqueness and edge-of-chaos behavior rather than trying to eliminate them entirely. Just as resummation landscapes naturally exhibit multiple optima when information is limited, AI loss landscapes will often support several distinct models that fit the data well. The goal is not to force uniqueness but to design objectives and evaluation pipelines that identify configurations that are robust, interpretable, and better aligned with truth across tasks.

Finally, the entire process should be seen as a dynamic interplay between $q(t)$ and τ . The training corpus and known benchmarks are the evolving record; the model parameters are a compressed representation; the loss function is a truth-favoring potential over parameter space. As in the resummation scheme, the art lies in shaping that potential so that its minima are not merely good fits to $q(t)$ but also plausible steps toward τ . The modest, one-dimensional examples of resummation with penalties offer a clear, analyzable setting in which to experiment with these ideas before extending them to the far more complex terrain of high-dimensional AI models.

10. Conclusion

The analysis of resummation with penalties, seen through a logistic lens, reveals far more than a clever way to tame divergent series. It exposes a small, tractable ecosystem where asymptotic analysis, catastrophe, self-reference, and notions of truth-favoring compression interact in a way that parallels much larger questions in complex systems and AI. The cost-functionals and their catastrophic variants do not simply improve numerical estimates of critical amplitudes; they carve out an optimization landscape whose structure embodies a theory of how partial, local imprints can be bent toward more truthful global behavior.

In this concluding section, we first summarize the logistic interpretation developed throughout the paper, then highlight how catastrophe, self-reference, and complexity appear in amplitude extraction, outline open questions at the interface of catastrophe theory, resummation, and logistics, and close with an outlook on bridging asymptotic analysis, complex systems thinking, and AI alignment.

10.1 Summary of the logistic interpretation of resummation with penalties

From a purely technical perspective, the scheme analyzed here starts with truncated asymptotic series and known or hypothesized power laws at infinity, then uses generalized Borel transforms, iterated roots, and cost-functionals to extrapolate amplitudes. The introduction of penalties dependent on amplitudes and their derivatives, with exponents m that control the degree of nonlinearity and non-smoothness, allows one to navigate the non-uniqueness of optimization solutions and to favor robust, accurate extrapolants.

The logistic interpretation reframes this entire apparatus in terms of $q(t)$ and τ . The truncated series, together with known exponents and external benchmarks, constitute $q(t)$: the evolving record of partial, local knowledge about a system's behavior. The cost-functionals define a potential landscape over the control parameter p , the perturbative order k , and the choice of

penalty exponent m . Minima of this landscape correspond to candidate embeddings of $q(t)$ into a more coherent, global structure. Those minima that exhibit stability across orders, consistency with external benchmarks, and insensitivity to small perturbations in p are interpreted as points closer to τ , the manifold of more truthful representations.

In this view, resummation with penalties is not neutral numerics; it is a prescription for bending an incomplete, local description of a system toward more globally consistent and truth-favoring configurations. The catastrophic penalties, in particular, sharpen this tendency by enforcing stronger consistency constraints and suppressing spurious minima, at the cost of introducing more complex optimization landscapes.

10.2 Catastrophe, self-reference, and complexity in amplitude extraction

A central result of the paper is that even one-dimensional optimization problems over a single parameter p can exhibit features classically associated with complex systems. The cost-functionals with $m \neq 2$ create landscapes with folds, cusps, and catastrophic “crash landing” minima. The presence of essential singularities in derivatives and abrupt changes in behavior as p crosses critical values are reminiscent of Thom’s catastrophe surfaces, compressed into a one-dimensional slice. This is particularly visible when catastrophic penalties transform otherwise smooth landscapes into ones with sharp valleys and competing minima.

At the same time, the iterated roots introduce self-reference and memory. Each approximant is defined in terms of the previous one; each amplitude sequence is fed back into the optimization via the cost-functional; and the control parameter p influences all orders at once. The resulting system behaves less like a direct mapping from series to amplitude and more like a dynamical process that revises its internal state in response to its own outputs and external constraints. This self-referential loop is what allows the scheme to adapt as k increases, refining its estimates and sometimes changing which penalty or parameter choice yields the best performance.

Taken together, catastrophe and self-reference show that amplitude extraction, when pushed beyond minimal prescriptions, becomes a small but genuine instance of complexity. Non-uniqueness, multiple attractors, and edge-of-chaos behavior are not pathologies to be eliminated; they are structural features of a system that is trying to reconcile limited data with strong asymptotic demands. The penalties shape this complexity, not by removing it, but by channeling it toward more physically meaningful solutions.

10.3 Open questions for catastrophe theory, resummation, and logistics

The explicit appearance of catastrophic structures in the cost-functionals suggests several open directions.

For catastrophe theory, a natural question is whether the different qualitative behaviors observed as m and k vary can be mapped onto standard elementary catastrophes in a higher-dimensional control space. Even though p is one-dimensional, the effective control space includes m , k , and weights such as λ , and the internal structure of the iterated roots introduces additional hidden coordinates. A more systematic classification could reveal which families of cost-functionals correspond to fold, cusp, or higher catastrophes, and what this implies for the stability and universality of the extracted amplitudes.

For resummation theory, there are open questions about optimal penalty design. The present work identifies a small family of robust performers, but it does not claim optimality in any formal sense. One might ask whether there exists, for particular classes of series or physical problems, a principled way to derive penalty exponents and weights from properties of the coefficients, such as their growth rate, sign structure, or known singularity structure. Another question is how to combine catastrophic and smooth components in a more systematic way, perhaps using multi-scale or adaptive schemes that vary m across orders or regions of p .

For logistics, the resummation scheme is a concrete testbed for ideas about $q(t)$, τ , and truth-favoring potentials. It raises deeper questions: how should one measure distance to τ in such a setting; how do different penalty designs correspond to different moral or epistemic stances about what makes an extrapolation more truthful; and can similar architectures be built for other kinds of partial imprints, such as empirical time series, spectral data, or high-dimensional simulations? The connection to Markov membranes and self-sustaining loops suggests that logistic structures may be identified and studied in many other corners of mathematical physics and beyond.

10.4 Outlook: bridging asymptotic analysis, complex systems, and AI alignment

The most speculative, but perhaps most promising, aspect of this work is its potential to bridge asymptotic analysis, complex systems theory, and AI alignment.

From asymptotic analysis, we learn how local series encode global behavior, and how sophisticated extrapolation schemes can reconstruct critical amplitudes from limited data. From complex systems, we recognize that such schemes inevitably generate rich internal dynamics, including multiple attractors, edge-of-chaos behavior, and emergent structure driven by self-reference and constraints. From AI alignment, we bring the question of how to design training

objectives and compression schemes that favor more truthful, coherent models over those that merely fit data.

Resummation with penalties can be read as a microcosm where these three strands meet. The truncated series and asymptotics play the role of a training corpus; the iterated roots are a simple architecture; the cost-functionals define a truth-favoring potential. Studying how different penalty designs affect accuracy, robustness, and complexity in this controlled setting offers conceptual guidance for designing loss functions and architectures in high-dimensional AI systems that better preserve τ -like structure.

The outlook is not that asymptotic resummation provides ready-made recipes for AI, but that it provides a mathematically transparent playground in which to experiment with the ideas of truth-favoring compression, catastrophic versus smooth constraints, and self-referential dynamics under partial information. As AI systems become ever more central to how $q(t)$ is generated, stored, and transformed, such playgrounds become valuable. They remind us that even in one dimension, the path from data to truth-like representation passes through landscapes shaped by catastrophe, self-reference, and carefully chosen penalties. The task ahead is to extend this insight to the high-dimensional, non-linear worlds where both physical theories and AI models now live, and to design potentials that bend their trajectories a little closer to τ .

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